Probabilistic Design of Slopes

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This paper discusses some improvements on the first-order second-moment (FOSM) probabilistic approach to slope design. The stability model by Morgenstern and Price is used for the formulation of the performance function, thus enabling the FOSM method to be applied to the probabilistic assessment of a general slip surface. A new solution scheme is also used herein for Morgenstern and Price's method. It does not require iterations for the calculation of the interslice forces and the derivatives of the performance function can be evaluated analytically. The reliability index β_{HL} defined by Hasofer and Lind is used as an index of safety measure. It has the advantage of being "invariant," that is, its value does not depend on the format of the performance function, a property considered lacking in the conventional reliability index. Reference is also made to the probabilistic modelling of soil profiles. The importance of the correlation structure of soil properties is highlighted and its effect on the reliability index β_{HL} is discussed.

Key words: slope stability, safety factors, reliability index, probability of failure, general slip surface, rigorous stability model.

Cet article traite d'améliorations apportées à l'approche probabilistique de premier-ordre second-moment (FOSM) pour le calcul des talus. Le modèle de stabilité de Morgenstern et Price est utilisé pour la formulation de la fonction de performance permettant ainsi d'appliquer la méthode FOSM à l'évaluation probabilistique d'une surface générale de glissement. Une nouvelle solution est également utilisée ici pour la méthode de Morgenstern et Price. Elle ne requiert pas d'itérations pour le calcul des forces entre les tranches et les dérivées de la fonction de performance peuvent être évaluées analytiquement. L'indice de fiabilité β_{HL} défrni par Hasofer et Lind est utilisé comme indice de la mesure de stabilité. Il a l'avantage d'être "invariant", c'est-à-dire que sa valeur ne dépend pas du format de la fonction de performance, une propriété qui est considérée comme manquante dans l' indice de fiabilité conventionnel. L'on fait référence également à la modélisation probabilistique des profils de sol. L'importance de la structure de corrélation des propriétés de sol est mise en lumière et son effet sur l'indice de fiabilité β_{HL} est aussi discuté.

Mots clés: stabilité des pentes, coefficients de sécurité, indice de fiabilité, probabilité de rupture, surface générale de rupture, modèle rigoureux de stabilité.

[Traduit par to revue]

INTRODUCTION

The safety of a slope is conventionally assessed by means of the factor of safety. It is seen that the factor of safety (FOS), F, is not a consistent measure of risk. Slopes with the same value of F may exhibit different risk levels depending on the variability of the soil properties. As the variability of soil properties changes with soil type and location, the choice of a suitable value of F would tend to rely mainly on local experience. Given the results of soil testing and other relevant information for the slope design, the question is often asked, "How large need the factor of safety be to be large enough?" The question is difficult to answer without regard to the failure probability of the slope.

As a result, there have been attempts in recent years to use a probabilistic approach for analyzing the safety of slopes. The variability in soil properties is taken into account and the failure probability is assessed. The major elements of such a probabilistic analysis are the statistical modelling of the soil properties in the field and the appropriate technique for calculating the risk of slope failure. The former has been addressed by Vanmarcke (1977a, 1984) and Li and White (1987a). This paper discusses some new developments of the technique for analyzing the reliability of slopes.

Reliability calculations are currently based on simplified stability models such as the ordinary method of slices and simplified Bishop's method. The reliability

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index β is used as an alternative risk measure to the usual factor of safety. These methods have limitations, since the use of simple stability models may not be sufficient for describing the performance of more complex slip surfaces. Moreover, the reliability index so defined gives a "variant" definition of risk measure. This presents a problem, as the values of β obtained from different formats of the performance function cannot be compared directly.

The approach presented herein adopts the stability model by Morgenstern and Price (1965) for the formulation of the performance function and the reliability index defined by Hasofer and Lind (1974). The latter is an "invariant" risk measure and hence all equivalent formats of the performance function yield the same reliability index. The use of the above approach is illustrated by a worked example and a case study.

PERFORMANCE FUNCTION

In a probabilistic approach, the failure - safety state of a slope can be described by the so-called performance function $G(X)$, where X denotes the vector of input parameters. The performance function is usually defined in such a way that failure of the slope is indicated by $G(X) < 0$ and safety by $G(X) > 0$. The hypersurface given by

$$G(X) = 0 \tag{1}$$

partitions the vector space X into two distinct regions, namely, the safety region of $G(X) > 0$ and the failure region of $G(X) < 0$. The hypersurface in Eq. (1) is called the limit state surface or boundary. The failure probability P_f is as follows:

$$P_f = \Pr\,(G(X) < 0) \tag{2}$$

In current approaches, $G(X)$ is formulated using simplified stability models such as the ordinary method of slices (Yucemen *et al.* 1973; Lee *et al.* 1983; Bao and Yu 1985) or the simplified Bishop' s method (Alonso 1976; Tobutt and Richards 1979; Tobutt 1982; Félio *et al.* 1984; Bergado and Anderson 1985). A rigorous model has not been used because the conventional solution scheme for the generalized procedure of slices does not provide a direct procedure for evaluating the interslice forces. As a result, the interslice forces need to be calculated by means of iterations, which also means that the derivatives of $G(X)$ required for the implementation of the probabilistic analysis must be evaluated numerically. Use of a rigorous model would therefore become much more involved.

In this paper, the rigorous stability by Morgenstern and Price is used. The original formulation by Morgenstern and Price is very complicated and difficult to use in the context of probabilistic analysis. A unified solution scheme developed recently by Li and White (1987c) for the formulation of Morgenstern and Price's method is therefore used herein. This enables the performance function to be defined explicitly without recourse to iteration for the calculation of the interslice forces. Furthermore, the derivatives of $G(X)$ can be evaluated analytically.

Some of the symbols and notations used in the derivation of $G(X)$ are defined in Figure 1. The subscript i denotes properties pertaining to the ith slice and the superscript ' represents effective stress properties. The symbol ˜ indicates the spatial average of individual slices, which are numbered from 1 to n in the positive x-direction. The following are the basic equations (Li and White 1987*b*, 1987*c*):

$$\begin{aligned} G_m(X) = \sum_{i=1}^{n} \{ [\tilde{c}_i'\Delta x_i + (\Delta W_i + \Delta T_i - \tilde{u}_i\Delta x_i)t_i] m_i y_{m_i} - [\Delta Q_i y_{Q_i} + (\Delta W_i + \Delta T_i) y_{m_i} \tan\alpha_i - \Delta T_i x_{m_i}] \} \\ -(E_b y_b - E_a y_a + T_b x_b + T_a x_a) \end{aligned} \tag{3}$$

$$G_f(X) = \sum_{i=1}^{n} \{ [\tilde{c}_i'\Delta x_i + (\Delta W_i + \Delta T_i - \tilde{u}_i\Delta x_i)t_i] m_i - [\Delta Q_i + (\Delta W_i + \Delta T_i)\tan\alpha_i] \} - (E_b - E_a) \tag{4}$$

$$\left.\begin{aligned} \Delta T_i &= \frac{E_{i-1} - \dfrac{T_{i-1}}{\lambda f(x_i)} + [\tilde{c}_i'\Delta x_i + (\Delta W_i - \tilde{u}_i\Delta x_i)t_i] m_i - (\Delta Q_i + \Delta W_i \tan\alpha_i)}{\dfrac{1}{\lambda f(x_i)} - t_i m_i + \tan\alpha_i}, \quad & i = 1,\mathrm{n} \\ \Delta T_\mathrm{i} &= T_b - \Delta T_{n-1}, & i = \mathrm{n} \end{aligned}\right\} \tag{5}$$

where $\tilde{c}_i'$ = the average effective cohesion over the base of the slice; $\tilde{\gamma}_i$ = the average density over the slice; t_i = the coefficient of shearing resistance, $\tan\phi_i'$; $\Delta W_i = \Delta P_i + p_i \Delta x_i + A_i \tilde{\gamma}_i$; $m_i = \sec^2\alpha_i \,/\, (1 + t_i \tan \alpha_i)$; $f(x)$ = the interslice force function[1] prescribing the variation of the inclination of the resultant interslice force; λ = a multiplication factor (to be determined).

The subscripts *m* and *f* in Eqs. (3) and (4) signify that the performance functions are based on the overall moment equilibrium and the force equilibrium condition respectively. Eq. (3) can be viewed as the subtraction of the total disturbing moment from the total resisting moment about point O. When the realization of the input parameters is such that the total resisting moment is less than the total disturbing moment (i.e., $G_m(X) < 0$), failure is implied. On the other hand, safety is indicated by $G_m(X) > 0$. A similar interpretation can be attributed to $G_f(X)$.

In deriving $G_m(X)$ and $G_f(X)$, the variation of $\tan\phi_i'$ is assumed uniform over the base of each slice, the average value of $\tan\phi_i'$ of the slice being represented by the value t_i at the midpoint of the slice. Theoretically, the moment arms x_{m_i} and y_{m_i} should also be regarded as random variables. The variability of x_{m_i} and y_{m_i} should, however, be small unless Δx_i is large. For practical purposes, they can be treated as deterministic quantities measured from the centres of the bases of the slices.

It can be seen that Eq.(5) involves only the interslice forces on the left of each slice. With the known conditions at the left $x = x_o$, all the interslice forces can be calculated explicitly in succession without the iteration required in conventional procedures. Because of this, the derivatives of the performance functions $G_m(X)$ and $G_f(X)$ can be evaluated analytically. A comprehensive list of formulae for the derivatives of $G_m(X)$ and $G_f(X)$ is given in the Appendix.

The failure probabilities inferred from $G_m(X)$ and $G_f(X)$ would generally be different. However, the value of λ can be adjusted so that the values of P_f obtained from both performance functions are equal. This concept has been used for obtaining the rigorous solution of the factor of safety (e.g., Morgenstern and Price 1965; Li and White 1987*c*).

Morgenstern and Price's method is chosen as the stability model in this work because it is commonly accepted as one of the accurate methods for slope stability analysis, and also because of its robustness and ability to incorporate some other models as special cases. However, other models based on the generalized procedure of slices could also be used in lieu of Morgenstern and Price's method. Using the unified solution scheme by Li and White (1987*c*), it is only necessary to replace the subroutines for the calculation of ΔT_i and its derivatives with respect to the basic input parameters. The solution procedure for other

[1] Some examples of this function are shown in Figure 2.

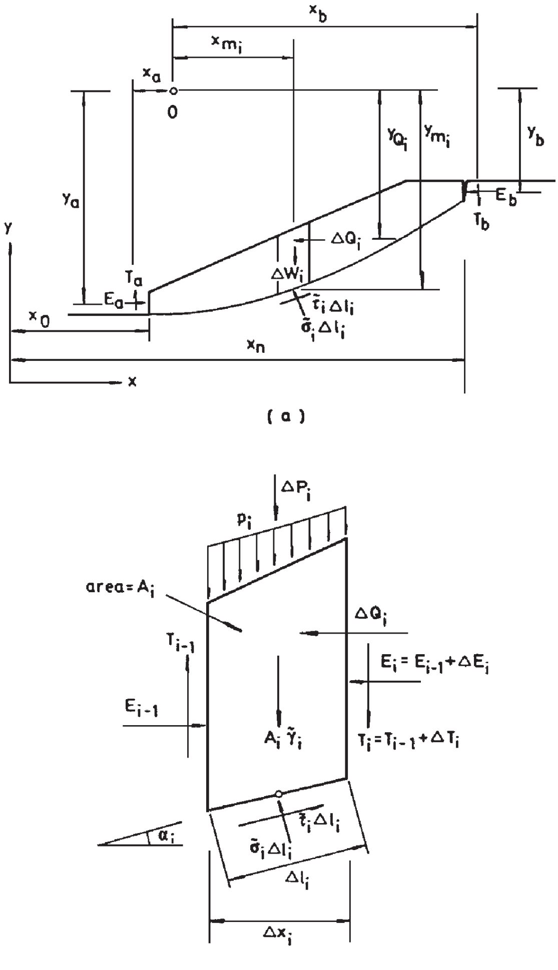

Figure 1. Definitions and notations used in the generalized procedure of slices: (a) external forces acting on slip surface; (b) forces acting on a slice

models is the same as that of Morgenstern and Price's method described herein. A list of the expressions of ΔT_i for different stability models can be found in Li and White (1987*c*).

RANDOM FIELD MODEL

Denote the value of a soil property at a point $t = (x, y, z)$ by $x(t)$. In general $x(t)$ can be decomposed into a trend component $g(t)$ and a random component $\varepsilon(t)$ with zero mean value, viz.,

$$x(t) = g(t) + \varepsilon(t) \tag{6}$$

Except for the sample points, the realization (i.e., the actual value) of a soil property at location *t* is not known and must therefore be regarded as a random

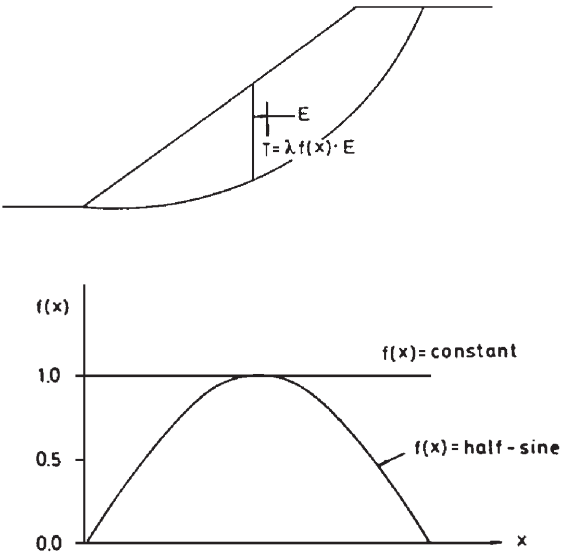

Figure 2. Interslice force function for Morgenstern and Price's method

variable. The realization of a soil property at location t is, in general, different from that at location t' even within a so-called homogeneous soil profile. To model the soil property correctly, one has to consider infinitely many random variables at all locations t. This important probabilistic nature of the soil property has not been properly recognized in much of the current literature on probabilistic slope design. Very often, the soil property is represented as a single random parameter. Examples of this are many (e.g., Matsuo and Kuroda 1974; A-Grivas *et al.* 1979; A-Grivas and Nadeau 1979; Tobutt and Richards 1979; He and Wei 1979; Webb 1980; Chowdhury 1981; Tobutt 1982, Lee *et al.* 1983; Sivandran and Balasubramaniam 1982; Bao and Yu 1985; Smith 1985; Young 1985). This has the implicit implication that the soil property is perfectly correlated over the soil profile, which also means that the realization of the property is the same at all locations. For example, if the cohesive strength at point A is 10 units, the strength at all other locations is also 10 units. If this is the true statistical representation of the soil profile, one sample will be adequate to establish the *in situ* property of the soil and there will be no uncertainty involved in the estimation of the soil property. Obviously, this is not the case for a real soil profile. As will be seen later, the assumption of perfect correlation also leads to gross overestimation of the failure probability of slopes.

The discussion here will be confined to a homogeneous random field, i.e., a soil profile with constant mean trend and statistical properties, but a more general treatment is given elsewhere (Li and White 1987*a*). In a homogeneous random field, the variation of $x(t)$ is described by means of the first- and second-order statistical moments (Vanmarcke 1977*a*, 1984).

$$E\{x(t)\} = g(t) = m = \text{constant} \tag{7a}$$

$$\text{var}\,\{x(t)\} = \text{var}\{\varepsilon(t)\} = \sigma^2 = \text{constant} \tag{7b}$$

$$\text{cov}\,\{x(t), x(t')\} = \text{cov}\,\{\varepsilon(t), \varepsilon(t')\} = \sigma^2\rho(v) \tag{7c}$$

E{ }, var { }, and cov { } are the expected value, variance, and covariance respectively; ρ( ) is the autocorrelation function (ACF), which depends only on the lag distance $v = (v_x, v_y, v_z) = |\, t' - t \,|$ between the points t and t'.

Eqs. (7a) and (7b) concern only the statistical property at a particular point, called the point property of the soil. On the other hand, Eq. (7c) describes the cross moment at two particular locations, called the cross point property of the soil. The point properties such as the coefficient of variation (COV) and the distribution are now well documented (Lumb 1966, 1970, 1974; Hooper and Butler 1966; Schultze 1975; Matsuo 1976; Krizek *et al.* 1977; Baecher *et al.* 1980; Webb 1980; Lee *et al.* 1983; Chowdhury 1984). However, information regarding the cross point properties is relatively sparse.

Soils generally exhibit plastic behaviour, although to a differing degree. As a result, the stability of a soil slope tends to be controlled by the averaged soil strength rather than the soil strength at a particular location along the slip surface. Also, the disturbing force acting on the slope is related to the average density of the soil. The spatial average of a soil property $x(t)$ is defined as

$$\tilde{x}_V = \frac{1}{V}\int_V x(t)dt \tag{8}$$

Where V can be the length L, area A, or the volume V of the spatial domain depending on the case. The mean, variance, and covariance of the spatial averages can be described by (Vanmarcke 1984; Li and White 1987a)

$$E\{\tilde{x}_V\} = m \tag{9a}$$

$$\text{var}\{\tilde{x}_V\} = \text{var}\{\tilde{\varepsilon}_V\} = \sigma^2(\Gamma^2(V)) \tag{9b}$$

$$\text{cov}\{\tilde{x}_V, \tilde{x}_{V'}\} = \text{cov}\{\tilde{\varepsilon}_V, \tilde{\varepsilon}_{V'}\} = \sigma^2(B(V,V')) \tag{9c}$$

where $\tilde{\varepsilon}_v$ denotes the spatial average of the random component $\varepsilon(t)$ over the domain V, $\Gamma^2(\,)$ and $B(\,)$ are called respectively the variance reduction and the covariance factor. $\Gamma^2(\,)$ is bounded by 0 and 1, that is to say, the variance of the averaged soil property is in general less than the variance of the point property. This effect arises because of the compensating effects as a result of spatial averaging. For instance, low values of strength at some locations are compensated by larger values at other locations within the spatial domain. In consequence, the fluctuation of the average strength and hence the variance are smaller. The reduction of variance due to spatial averaging depends on the rate of decay of the ACF. For ACF's that decay rapidly with lag distance (i.e., soil properties with a small correlation distance or scale of fluctuation), the variance reduction is significant and vice versa. For some natural soils, the correlation distance of soil properties is small and the soil properties become largely uncorrelated for lag distances greater than 1-2 m (Li and White 1987*a*). The reduction of variance due to spatial averaging can therefore be appreciable without the averaging dimension being very large. Although it may be very discomforting to realize that a COV of greater than 40% (point property) is not uncommon for the undrained shear strength of soils (Alonso 1976), the variability of the average shear strength, which governs the performance of the slope, is usually much less than that of the point variability.

Eq. (9) assumes that the mean value of the soil property is known. In practice, it has to be estimated from samples taken at different locations t_i of the field. Thus

$$E\{x(t)\} \rightarrow \overline{m} = \frac{\sum_{i=1}^{k} x(t_i)}{k} \tag{10}$$

where the arrow means "estimated by" and k is the total number of samples. The spatial average of the soil property will then be estimated by the sample spatial average $\bar{x}_v$, defined by

$$\bar{x}_V = \frac{1}{V}\int_V [\overline{m} + \varepsilon(t)]\, dt \tag{11}$$

The expected value of $\bar{x}_v$ is

$$\begin{aligned} E\{\bar{x}_V\} &= \frac{1}{V}\int_V E\{\overline{m} + \varepsilon(t)\}\, dt \\ &= \frac{1}{V}\int_V E\{\overline{m}\}\, dt \\ &= \frac{1}{V}\int_V m\, dt \\ &= m \end{aligned} \tag{12}$$

Therefore, $\bar{x}_v$ is an unbiased estimator for the mean value of the spatial average $\tilde{x}_v$. The variance of $\bar{x}_v$ is expressed as

$$\begin{aligned} \operatorname{var}\{\bar{x}_V\} &= E\{\bar{x}_V - m\}^2 \\ &= E\left\{\frac{1}{V}\int_V [\overline{m} - m]\, dt + \frac{1}{V}\int_V \varepsilon(t)\, dt\right\}^2 \\ &= E\{(\overline{m} - m) + \tilde{\varepsilon}_V\}^2 \end{aligned} \tag{13}$$

For simplicity, the correlation of the soil property at the sample points t_i and the soil property within the spatial domain V is neglected and Eq. (13) becomes

$$\begin{aligned} \operatorname{var}\{\bar{x}_V\} &= \operatorname{var}\{\overline{m}\} + \operatorname{var}\{\tilde{\varepsilon}_V\} \\ &= \operatorname{var}\{\overline{m}\} + \sigma^2(\Gamma^2(V)) \end{aligned} \tag{14}$$

If t_i are sufficiently wide apart, the correlation of the soil property among the sample points can also be neglected and the variance of the sample mean can be approximated by

$$\operatorname{var}\{\overline{m}\} = \frac{\sigma^2}{k} \tag{15}$$

Therefore, var $\{\bar{x}_V\}$ can be estimated by

$$\operatorname{var}\{\bar{x}_V\} \rightarrow s^2\left\{\frac{1}{k} + \Gamma^2(V)\right\} \tag{16}$$

where s^2 is the sample variance of the test results. Eq. (16) indicates that the overall uncertainty for the estimated average soil property consists of two parts - the inherent variability associated with the point-to-point variation of the soil property in the field and the sampling uncertainty associated with the estimation of the trend component. The latter is sometimes not recognized in current literature. Similarly, the covariance of the sample spatial averages is given by

$$\operatorname{cov}\{\bar{x}_V, \bar{x}_{V'}\} \rightarrow s^2\left\{\frac{1}{k} + B(V,V')\right\} \tag{17}$$

FACTOR OF SAFETY APPROACH

The safety of slopes is often assessed by means of the factor of safety, F, calculated using a deterministic procedure and (typically) mean values of input parameters. Methods are now available whereby the value of F can be calculated very efficiently to the required precision (Li and White 1987c). Despite the simplicity of the FOS approach, it has several shortcomings, which can be discerned by means of a simple example of a cohesive slope as shown in Figure 3. The first disadvantage of the FOS approach is the "variance" of the definition of F, that is, the value of F depends on how F is defined (Höeg and Murarka 1974; Chae 1967). With the notations given in Figure 3, the FOS is usually defined as

$$F = \frac{\bar{c}LR}{W_1 d_1 - W_2 d_2} \tag{18}$$

where $\bar{c}$ is the mean cohesive strength of the soil. However, some engineers prefer to treat the soil mass W_2 as contributing to the stability of the slope and define the FOS as

$$F = \frac{\bar{c}LR + W_2 d_2}{W_1 d_1} \tag{19}$$

There can be a substantial difference for the computed value of F depending on whether the term W_2d_2 appears in the numerator as part of the resisting moment or in the denominator as part of the overturning moment, as is indicated in Table 1 for some actual slope designs. The same argument applies to the way pore-water pressure is treated in slope stability analysis. The pore-water pressure term can appear either in the numerator or the denominator depending on whether it is treated as a loading to the system or as a reduction to the strength term.

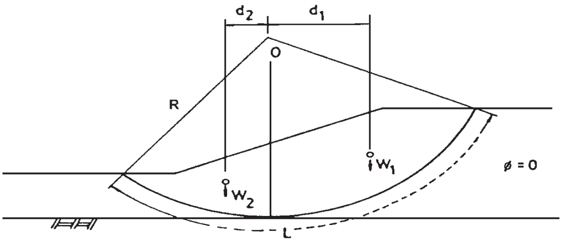

Figure 3. Details of a cohesive slope

The second undesirable property of the FOS approach is that it is not a consistent measure of structural safety. Table 2 shows the failure probability of the slope in Figure 3 assuming Gaussian distributions and independence for the average shear strength and soil density. V_R and V_S in the table denote respectively the COV of the resisting and the disturbing moment. A wide range of values of P_f can be obtained for the same value of F. Therefore, specifying a constant value of FOS cannot ensure a consistent risk level of slopes. As a corollary, it is impossible to say how much safer a slope becomes as the FOS is increased.

Table 1. Variation of factor of safety (after Chae 1967)

Case	Factor of safety based on:	
	Eq. [18]	Eq. [19]
1	1.57	1.22
2	1.70	1.24
3	0.63	0.75
4	0.81	0.87
5	0.74	0.83
6	0.67	0.70
8	2.00	1.67

Table 2. Variation of P_f with variability of soil properties for a constant FOS of 1.5 (after Lumb 1983)

V_R	V_S	P_f
0.2	0.2	8.3×10^{-2}
0.2	0.05	5.0×10^{-2}
0.1	0.2	2.3×10^{-2}
0.1	0.05	7.8×10^{-4}
0.05	0.2	9.6×10^{-3}
0.05	0.05	1.4×10^{-8}

Figure 4 shows the variation of P_f with F assuming that the soil density is constant and the cohesive shear strength is a Gaussian variate with a typical value of 0.3 for the COV. The variance reduction factor for the length of the slip surface is represented by $\Gamma^2(L)$. As indicated in this example, the value of P_f is sensitive to the value of F within the typical range of design FOS (1.2 – 1.5) when the variance reduction factor is smaller than about 0.3, which is not uncommon for real slopes.

Because of the above drawbacks, the factor of safety of a slope is not a satisfactory risk measure. A partial FOS approach has been proposed as an alternative to the overall FOS approach (Brinch Hansen 1967; Lumb 1970; Meyerhof 1970, 1984). However, it cannot eliminate the shortcomings of the FOS approach.

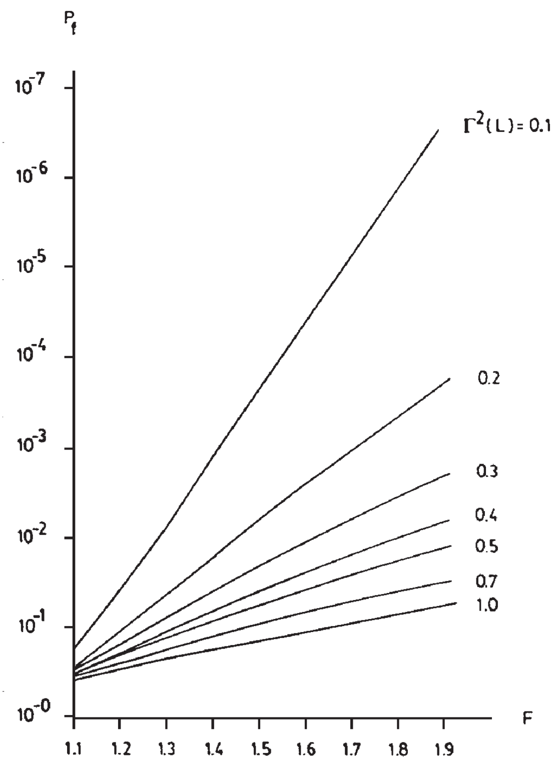

Figure 4. Variation of failure probability with factor of safety (after Li and White 1987b)

FIRST-ORDER SECOND-MOMENT (FOSM) METHOD

Since the early 1970's, there has been a trend towards use of a first-order second-moment probabilistic approach for analyzing the reliability of slopes. In this approach, the performance function is linearized by means of a first-order Taylor's series approximation and the random parameters are characterized by their first two moments (hence the name). As information regarding the joint distribution of soil properties is generally not available, more rigorous probabilistic approaches such as the advanced FOSM method in structural reliability analyses (e.g., Paloheimo and Hannus 1974; Rackwitz and Fiessler 1978; Chen and Lind 1983) are difficult to use.

In current probabilistic analyses, the reliability index β defined by

$$\beta = \frac{\mu_G}{\sigma_G} \tag{20}$$

in which μ_G and σ_G denote the mean value and standard deviation of the performance function $G(X)$ respectively, is often used as an alternative risk measure to the conventional FOS (e.g., Yucemen *et al.* 1973; Alonso 1976, Vanmarcke 1977*b*, 1980; Bergado and Anderson 1985). The use of β as a safety measure is based on the following observation:

$$\begin{aligned} P_f &= \Pr\,(G(X) < 0) \\ &= \Pr\left(\frac{G(X) - \mu_G}{\sigma_G} < -\frac{\mu_G}{\sigma_G}\right) \\ &= \Pr\,(Z < -\beta) \\ &= \int_{-\infty}^{-\beta} \psi(z)\,dz \\ &= \Psi(-\beta) \end{aligned} \tag{21}$$

Here, Z is a standardized variable of $G(X)$; $\psi(z)$ and $\Psi(z)$ are respectively the probability density function (PDF) and the cumulative distribution function (CDF) of Z. As $\Psi(\ )$ is a non-decreasing function, a one-to-one correspondence exists between the failure probability and the reliability index β. All the uncertainties of the random variables have been suitably condensed into a single reliability index β. Provided that the reliability indices for two similar slopes are equal, they will have a similar risk level, although the variability of the random variables may be different in the two cases.

Knowing the first two moments of $G(X)$ is not sufficient to define the PDF of Z or $G(X)$. A Gaussian or lognormal distribution is usually assumed for $\Psi(\ )$.

Although the reliability index β is a consistent index of risk measure, it is not "invariant". Table 3 shows the reliability indexes for different formats of the performance function using the FOSM method. R and S in the table represent respectively the total resisting force and the disturbing force acting on the slope. An examination of Table 3 shows clearly the property of "variance" of the reliability index β.

Table 3. Risk format and reliability index β

$G(X)$	β
$R - S$	$\dfrac{F-1}{\sqrt{F^2 V_R^2 + V_S^2}}$
$\dfrac{R}{S} - 1$	$\dfrac{F-1}{F\sqrt{V_\mathrm{R}^2 + V_S^2}}$
$\ln\dfrac{R}{S}$	$\dfrac{\ln \mathrm{F}}{\sqrt{V_\mathrm{R}^2 + V_\mathrm{S}^2}}$

Note:

$F = \overline{R}/\overline{S}$.

To circumvent this problem, Hasofer and Lind (1974) proposed an invariant definition for the reliability index. In this format, all the random variables X are transformed into a standardized parameter space Z by means of an orthogonal transformation such that

$$E\{Z_i\} = 0; \quad \mathrm{var}\,\{Z_i\} = 1; \quad \mathrm{cov}\,\{Z_i, Z_j\} = 0 \tag{22}$$

Hasofer and Lind (1974) defined the reliability index as the minimum distance between the origin and the limit state surface in the transformed parameter space Z. To distinguish the reliability index defined in Hasofer and Lind's sense from that defined by Eq. (20), the former will be denoted by β_{HL}. The property of "invariance" for the reliability index β_{HL} is clear from its definition. As an example, let us consider the different formats of $G(X)$ in Table 3. For the first format, the limit state surface is given by

$$R - S = 0 \tag{23}$$

The limit state surface for the second format is described by $(R/S) - 1 = 0$, which, after simplification, yields the same equation as Eq. (23). The same is true for the third format. Since the limit state surfaces of these formats are the same, the minimum distance between these surfaces to the origin in the Z-space, and hence β_{HL}, will be equal.

The point Z_d on the transformed limit state surface with the minimum distance from the origin in the Z-space will be called the design point in the Z-space. Likewise, the inverse mapping of Z_d in the X-space will be called the design point in the X-space, designated as X_d.

If the random parameters are jointly Gaussian and the performance function is linear, the failure probability is simply related to the reliability index β_{HL} by (Leporati 1979)

$$P_f = \Phi(-\beta_{HL}) \tag{24}$$

where $\Phi(\ )$ is the CDF of a standard Gaussian variate. However, Eq. (24) is commonly used for other cases of nonlinear performance functions and (or) non-Gaussian parameters to give a rough estimate of the magnitude of the failure probability. It can also be proved that if the performance function is linear, the reliability indices defined in the conventional sense β and in Hasofer and Lind's sense β_{HL} are equal (Li and White 1987b).

Although the use of β_{HL} has gained popularity in structural reliability analyses, it is less commonly used in geotechnical reliability analyses. It is considered that the reliability index β_{HL} is preferable to the conventional reliability index β. The former has the advantage of being an invariant index. Values of β_{HL} can be compared directly even though they may be derived from different formats of the performance function. Therefore, as far as codified design is concerned, a suitable minimum value of β_{HL} can be specified. The exact format of $G(X)$ need not be stipulated in the code as a result of the invariance of β_{HL}.

The reliability index β_{HL} for a slope can be calculated using the iterative algorithm by Parkinson (1978). Denote the random variables collectively by $X = (X_1, X_2, \ldots, X_l)$ where X_i can be $\tilde{c}_i'$, $\tilde{\gamma}_i$, t_i etc. If $X^{(j)}$ represents the jth estimate of the design point in the X-space, X_d, satisfying the limit state equation $G(X^{(j)}) = 0$, the $(j+1)$th estimate can be obtained using the iterative equation

$$X^{(j+1)} = \overline{X} + V_x \nabla G \left\{ \frac{(x^{(j)} - \overline{X})^T \nabla G}{\nabla G^T V_x \nabla G} \right\} \tag{25}$$

where

$$\overline{X} = (\mu_1, \mu_2, \ldots, \mu_i, \ldots, \mu_\ell)^T \tag{26}$$

$$\nabla G = \left[\frac{\partial G}{\partial X_1}, \frac{\partial G}{\partial X_2}, \ldots, \frac{\partial G}{\partial X_i}, \ldots, \frac{\partial G}{\partial X_\ell} \right]_j^T \tag{27}$$

in which $\overline{X}$ is the mean vector, μ_i represents the mean value of X_i, and V_x is the covariance matrix for X. The subscript j in Eq. (27) indicates that the partial derivatives are taken at the jth trial point $X^{(j)}$. The superscript T means the transpose of a matrix. It should be noted that the $(j+1)$th estimate obtained using Eq. (25) does not necessarily satisfy the limit state equation $G(X) = 0$. Therefore, it must be adjusted before it can be input into Eq. (25) for the next iteration. The adjustment can conveniently be done by choosing all of the parameter values but one to be the same as those of the unadjusted vector $X^{(j+1)}$; the remaining parameter value can be derived from the limit state equation. In this paper, the average cohesion at the nth slice is chosen arbitrarily to be the parameter for adjustment. The reliability index β_{HL} for the jth trial estimate $X^{(j)}$ is given by (Parkinson 1978)

$$\beta_{HL}^{(j)} = \left| (X^{(j)} - \overline{X})^T \nabla G \left\{ \nabla G^T V_X \nabla G \right\}^{1/2} \right| \tag{28}$$

where | | denotes absolute value. Convergence of the iterative procedure can be checked by whether the difference between successive values of $\beta_{HL}^{(j)}$ or $X^{(j)}$ is small. The partial derivatives of the performance function required for the implementation of the iterative algorithm are given in the Appendix.

It should be remembered that the exact mean values μ_i are not known. In practice, they are estimated using the sample mean values $\overline{X}_i$. Consequently, the variance and covariance of the soil properties have to be calculated using Eqs. (16) and (17) for the generation of the covariance matrix V_X.

Note that the performance functions $G_m(X)$ and $G_f(X)$ are linear with respect to $\tilde{c}_i$, $\tilde{\gamma}_i$, $\tilde{u}_i$ etc. and are only nonlinear with respect to t_i. Therefore, for $\phi = 0$ analyses, the algorithm will always converge to the design point X_d after the first iteration independent of the initial values used for the parameters. For c-ϕ slopes, the performance function is nonlinear with respect to t_i. A good initial estimate for the design point X_d can be obtained using the following procedure. Initially, t_i is assumed to be deterministic (i.e., the variance of t_i is taken to be zero) and is assigned a value equal to its mean value. By iterating once using Eq. (25), the design point for the "conditioned" performance function is obtained. This conditioned design point serves as a robust starting point for the general iteration.

EXAMPLE PROBLEMS

In this section, the implementation of the probabilistic approach using β_{HL} will be depicted by means of an illustrative example. The method is then applied to a case study of the Selset landslide reported in Skempton and Brown (1961). The assumptions used in the following discussions are mentioned first.

1. In the analysis, only $\tilde{c}_i'$, $\tilde{\gamma}_i$, t_i and $\tilde{u}_i$ are taken as random variables. Other loads (E_a, p_i, ΔP_i, ΔQ_i, *etc*) are taken as zero.
2. Little has been published in the current literature on the joint PDF of soil properties. Under controlled conditions such as constant soil density and moisture content, Matsuo and Kuroda (1974) observed a strong negative correlation between

the strength components c and t. However, for natural soils, evidence (Lumb 1970; Schultze 1975) shows almost zero correlation between the strength parameters. For practical purposes, the strength components c' and t can be regarded as independent. The assumption of mutual independence will simplify the computation and also err on the conservative side. The influence of the variability of soil density on P_f of slopes is usually small (Alonso 1976). This is due to the fact that the variability of soil density is small. Furthermore, the averaging dimension for γ_i is large. The variability of the average soil density is further reduced. In consequence, the cross correlation of γ_i with c' and t, which is of secondary importance, can be neglected without incurring significant errors.

3. The soil properties are modelled as random fields. The variance and covariance for the sample spatial averages for $\bar{c}'_i$ and $\bar{\gamma}_i$ are evaluated using the formulae given in Li and White (1987*a*). To be consistent with the assumption used in deriving the performance function $G_m(X)$ and $G_f(X)$, t_i is represented by the point property at the centres of the bases of the slices.
4. Little is known about the statistical properties, especially the correlation structure, of pore-water pressure. It is speculated that $\tilde{u}_i$ may consist of two random components with different scales of fluctuation. The first component is governed by the changes in the regional water system and therefore a larger scale of fluctuation is expected for this component. The second component is associated with the local variation of hydrological properties of the soil such as the permeability and the infiltration capacity and a smaller scale of fluctuation is expected. In theory, the pore-water pressure can also be modelled as random fields. Over the past decade or so, some conceptual models have been proposed along these lines, for example, Smith and Freeze (1979*a*, 1979*b*), Chirlin and Dagan (1980), Andersson and Shapiro (1983), Kitanidis and Vomvoris (1983), and Bergado and Anderson (1985). The Monte Carlo simulation is usually used as the basic technique for generating the statistical properties of the pore-water pressure. This procedure will involve repeated calculations using finite element programs, for instance, and will no doubt be costly to perform. More work needs to be done before these theoretical models can be put into practical use. In this work, a simple model is used. The pore-water pressure ratio r_u describes $\tilde{u}_i$:

$$\tilde{u}_i \Delta x_i = r_u (A_i \bar{\gamma}_i) \tag{29}$$

where $\bar{\gamma}_i$ is the mean soil density. Pending more information on the correlation structure of pore-water pressure, the pore-water ratio r_u is assumed to be perfectly correlated within the slope, i.e., r_u is regarded as a single variable. Furthermore, the cross correlation of r_u with other soil properties is neglected for simplicity.

Because of the above assumptions, the covariance matrix of X has the form

$$V_x = \begin{Bmatrix} V_{\tilde{c}'} & 0 & 0 & 0 \\ 0 & V_t & 0 & 0 \\ 0 & 0 & V_{\tilde{\gamma}} & 0 \\ 0 & 0 & 0 & \sigma^2_{r_u} \end{Bmatrix} \tag{30}$$

where $\tilde{c}_i'$ denotes the collection of $\tilde{c}_i'$ and t the collection of t_i, etc.

The geometry of the worked example is shown in Figure 5 and the input parameters are given in the following table:

Parameter	Mean	COV
c'	18 kPa	20%
γ	18 kN/m^3	5%
t	tan 30°	10%
r_u	0.2	10%

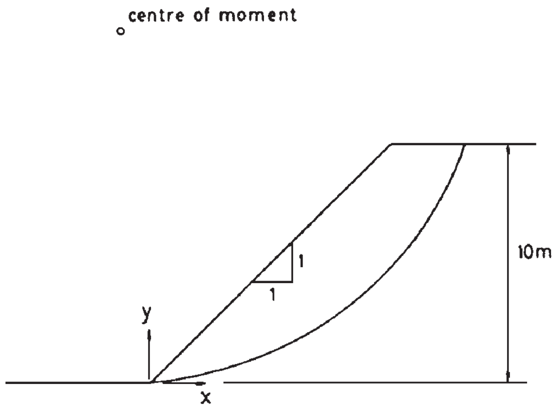

Figure 5. Geometry of the illustrative example

A number of two-dimensional separable ACF's as listed in Table 4 are used. The parameter δ represents the scale of fluctuation in the respective directions. It is a measure of the spatial extent within which soil properties show strong correlation (Vanmarcke 1984). A large value of δ implies that the soil property is highly correlated over a large spatial extent, resulting in a smooth variation within the soil profile. On the other hand, a small value of δ will indicate that the fluctuation of the soil property is large. Although different values of δ can be used for different soil properties, they are assumed to be equal in this example. Unless stated otherwise, results presented below are based on type I ACF and the scales of fluctuation in the x- and y-directions are assumed to be equal, i.e., $\delta_x = \delta_y = \delta$. The number of samples k is taken to be 8. Ten slices are used throughout and the effect of tension cracks is neglected. Two different interslice force functions as shown in Figure 2 are used in the analysis.

Figure 6a shows the typical variation of the reliability index β_{HL} with λ for the slip surface shown in Figure 5. β^m_{HL} and β^f_{HL} represent the reliability index β_{HL} based on the performance functions $G_m(X)$ and $G_f(X)$ respectively. The intersection point of the curves gives the so-called "rigorous" solution, β^r_{HL}. The corresponding value of λ is designated as λ_{mf}. At this point, the values of the failure probability predicted by the two performance functions are equal. A few interesting points need to be made.

1. For all the data points in Figure 6, a maximum of six iterations are found to be sufficient to give a tolerance of 10^{-8}. The rate of convergence is extremely fast.
2. The variation of β^m_{HL} with λ is much smaller than that of β^f_{HL}, as typified by Figure 6. Similar observations have also been reported elsewhere for the variation of FOS with λ (e.g., Li and White 1987*c*). Therefore, the value of β^m_{HL} based on $G_m(X)$ and a "reasonable" value of λ will usually be sufficiently close to the "rigorous" solution. β^f_{HL} is not recommended for use as an approximation to the rigorous solution β^r_{HL} because of its sensitivity to λ .

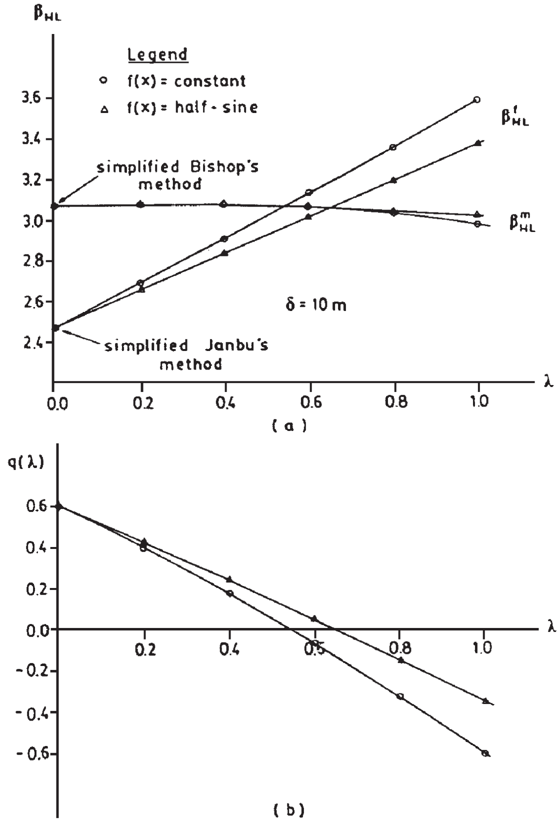

Figure 6. (a) Variation of β_{HL} with λ; (b) variation of $q(\lambda)$ with λ

To calculate β^r_{HL} , λ has to be adjusted to achieve the equality of β^m_{HL} and β^f_{HL}. This can conveniently be done using the procedure of inverse rational approximation. Define a function

$$q(\lambda)=\beta^m_{HL}-\beta^f_{HL} \tag{31}$$

The rigorous solution will then be given by the root of the equation $q(\lambda) = 0$. Figure 6*b* shows the variation of the function $q(\lambda)$ for two different interslice functions. The smoothly varying monotonic variations shown in the figure are typical. It is therefore possible to define an inverse for $q(\lambda)$, designated as $\lambda = \omega(q)$. Initially, the values $q_i = q(\lambda_i)$, i = 1, m are calculated for m different values of λ. The inverse function $\omega(q)$ can then be approximated by a rational polynomial $\eta_m(q)$ expressed in the following forth of a continued fraction:

$$\omega(q) \approx \eta_m(q) = a_1 + \cfrac{q-q_1}{a_2 + \cfrac{q-q_2}{a_3 + \cfrac{\ddots}{+\cfrac{q-q_{m-1}}{a_m}}}} \tag{32}$$

The coefficients a_i in Eq. (32) can be computed using the procedure described in Table 5.

Since the rigorous solution is given by $\omega(0)$, an approximate solution for λ_{mf} based on a m-term continued fraction, designated as λ_{m+1}, can be obtained using $\eta_m(0)$. The corresponding value of $q(\lambda_{m+1})$ is then computed. With this new set of interpolation points (λ_{m+1}, q_{m+1}), a better approximation to the inverse function can be obtained by lengthening the continued fraction to (m+1) terms. The above procedure is then repeated until the required tolerance for $q(\lambda)$ is achieved. The algorithm can be invoked using two starting trial points. As it turns out, the procedure of inverse rational approximation gives fast convergence for the calculation of the rigorous solution as typified by the results in Table 6 for the slip surface shown in Figure 5.

Figure 7 shows a comparison of the value of β^r_{HL} obtained from different interslice force functions $f(x)$ and ACF's. It can be observed that the values of β^r_{HL} obtained from the constant and half-sine interslice force functions are essentially the same, even though the two functions are drastically different. The insensitivity of Morgenstern and Price's method to the assumption used for the interslice forces is well established for the FOS calculations (e.g., Morgenstern and Price 1965; Fredlund and Krahn 1976). It is interesting to know that the same characteristic still exists for the reliability index β^r_{HL}. Figure 7 also indicates that the reliability index β^r_{HL} is not very

Table 4. Types of autocorrelation function

Type	Autocorrelation function
I. Simple exponential	$\exp\left\{-2\left(\frac{v_x}{\delta_x}+\frac{v_y}{\delta_y}\right)\right\}$
II. Square exponential	$\exp\left\{-\pi\left(\frac{v_x^2}{\delta_x^2}+\frac{v_y^2}{\delta_y^2}\right)\right\}$
III. Second-order autoregressive model	$\exp\left\{-4\left(\frac{v_x}{\delta_x}+\frac{v_y}{\delta_y}\right)\right\}\left(1+\frac{4v_x}{\delta_x}\right)\left(1+\frac{4v_y}{\delta_y}\right)$
IV. Cosine exponential	$\exp\left\{-\left(\frac{v_x}{\delta_x}+\frac{v_y}{\delta_y}\right)\right\}\cos\frac{v_x}{\delta_x}\cos\frac{v_y}{\delta_y}$

Table 5. Coefficients of rational polynomial

a_1	a_2	a_3	a_4
$a_1 = \lambda_1$			
$a_{21} = \lambda_2$	$a_2 = \frac{q_2 - q_1}{a_{21} - a_1}$		
$a_{31} = \lambda_3$	$a_{32} = \frac{q_3 - q_1}{a_{31} - a_1}$	$a_3 = \frac{q_3 - q_2}{a_{32} - a_2}$	
$a_{41} = \lambda_4$	$a_{42} = \frac{q_4 - q_1}{a_{41} - a_1}$	$a_{43} = \frac{q_4 - q_2}{a_{42} - a_2}$	$a_4 = \frac{q_4 - q_3}{a_{43} - a_3}$
...	...	...	...

Table 6. Adjustment of λ using inverse rational approximation for Morgenstern and Price's method

λ	β^m_{HL}	β^f_{HL}	$q(\lambda)^{\dagger}$
0.3*	3.657788	3.257851	0.39994
1.0*	3.587350	3.987004	–0.39965
0.65012	3.636614	3.631897	0.00472
0.65425	3.636209	3.636209	–0.0000007

* Initial trial values.

† The interslice function is taken to be half-sine function and δ = 5 m.

sensitive to the type of ACF's used especially when the scale of fluctuation is small compared with the dimension of the slip surface. It has an important implication, as the exact form of the ACF is difficult to be estimated without a large number of samples. The scale of fluctuation has already captured the essential correlation structure of the soil properties. Because of its simplicity, type I (simple exponential) ACF is recommended for general use.

Figure 8 shows the variation of β^r_{HL} with the scales of fluctuation for the slip surface shown in Figure 5. A unit function is taken for $f(x)$. It can be seen that β_{HL} and hence the failure probability are very sensitive to the scales of fluctuation. Therefore, more attention must be paid to the estimation of this important parameter in soil investigation. Over the years, numerous analyses have appeared in the literature in which the soil properties in the field are assumed to be single random variables. This is equivalent to saying that the scale of fluctuation is infinitely large. As indicated in Figure 8, it will yield a smaller value of β^r_{HL} and the failure probability will therefore be grossly overestimated.

Figure 9 shows the locations of the critical slip circle with minimum FOS and β^r_{HL}. The function $f(x)$ is taken to be a half-sine function. The figure indicates that the locations of the critical slip circles with minimum FOS and β^r_{HL} are not coincident, but very close to each other. A noncircular slip surface,

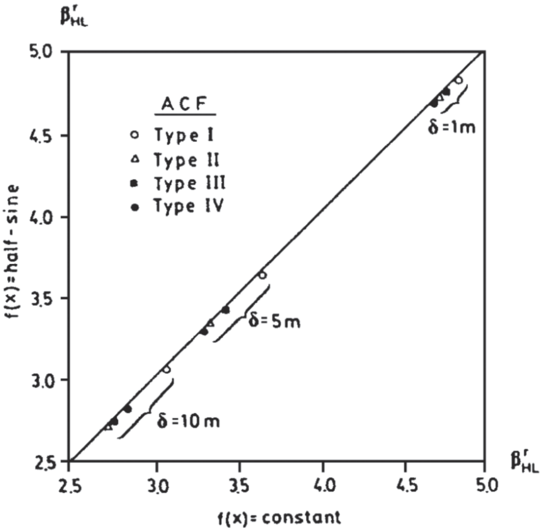

Figure 7. Comparison of β_{HL} for different interslice force and autocorrelation functions

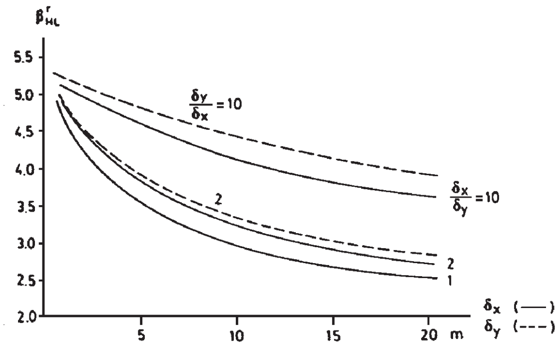

Figure 8. Variation of β_{HL} with scales of fluctuation

defined by four straight-line segments, has also been used to define the slip surface. The location of the critical noncircular slip surface with minimum β^r_{HL} is also shown in Figure 9. For this case of a homogeneous slope, the locations of the critical circular and noncircular slip surfaces are close, as are the corresponding value of β^r_{HL}. The observation is similar for other interslice force functions and ACF's. As the evaluation of β^r_{HL} requires much more effort than that of FOS, it is preferable to locate the critical slip surface with the minimum value of FOS first. The slip surface is then used as the initial trial surface for the general search for the critical slip surface with minimum β^r_{HL}.

Next, we consider a case study of the Selset landslide reported in Skempton and Brown (1961). The slip was within a deposit of nonfissured overconsolidated boulder clay. No significant variation of mean soil properties was observed within a depth of 60 ft (18.3 m). The soil profile could therefore be modelled as a homogeneous random field. The slope was 42 ft (12.8 m) high with an inclination of 28°.

Eight samples were taken at different locations of the slope. For each sample, at least three specimens were prepared for drained triaxial tests. Because of the proximity in the field, the soil properties of the test specimens from each sample will be highly correlated. Because of this, the mean soil property determined from test specimens of a sample would constitute effectively one single sample in the statistical sense. Hence, a value of 8 is used for the sample size k in this case. Since the sample locations were far apart in the field, the soil properties determined from each sample can be regarded as independent. The variance and covariance of the sample spatial average can therefore be evaluated using Eqs. (16) and (17).

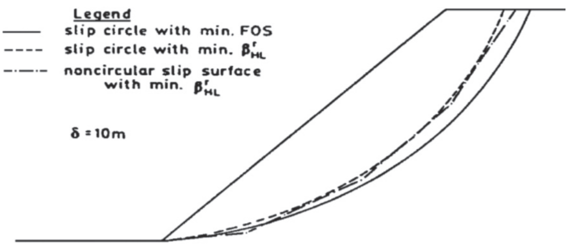

Figure 9. Locations of critical slip surfaces

A summary of the test results is given in Skempton and Brown (1961), from which the following input parameters are derived:

Parameter	Mean	COV
c'	180 lb/sq ft (8.6 kN/m^2)	30%
γ	139 lb/cu ft (21.8 kN/m^3)	0.7
ϕ	32°	7%

Using a first-order Taylor's series approximation, the mean value and COV of t (i.e., tan ϕ) are given as tan 32° and 9%. A mean value of 0.45 was suggested by Skempton and Brown (1961) as a suitable value for the pore-water pressure ratio r_u of the slope. A judgemental value of 10% is assumed here for the COV of r_u.

Figure 10 shows the variation of the failure probability with the height of the slope. The reliability index β_{HL} is based on $G_m(X)$ and a value of 0.6 for λ. The failure probability P_f is calculated using Eq. (24) and the minimum reliability index β_{HL} associated with the critical slip circle. A toe failure is assumed throughout. It is also assumed that the mean value and variance of r_u are unaffected by a change in the height of the slope. Since the scale of fluctuation of the soil properties is not known, two values of δ are used – 5 ft (1.5 m) and 15 ft (4.6 m). The results are plotted as solid lines in Figure 10. For the actual slope height of 42 ft (12.8 m), the failure probability of the slope is high (> 0.5). This is to be expected as the failure of the slope had indeed occurred. As the slope height decreases, the factor of safety increases and the difference in P_f given by the two values of δ becomes more pronounced.

Assuming that the COV of the soil properties remains unchanged, the slope is reanalyzed using a value of 30 for the sample size k. Since the sample size is now larger, the sampling uncertainty is reduced, resulting in a smaller value of P_f. Note that the increase in reliability of the slope due to an increase in sample size is greater for the case of δ = 5 ft (1.5 m). In fact, it is generally true that increasing the sample size to reduce the failure probability is more effective for soils with a smaller value of δ than for those with a larger value of δ.

In the design of soil slopes, results like Figure 10 can be obtained based on the prior knowledge of the soil properties. This kind of information would be very useful in the design stage of identifying the critical parameters to which more attention should be paid and for determining a suitable sample size for soil testing.

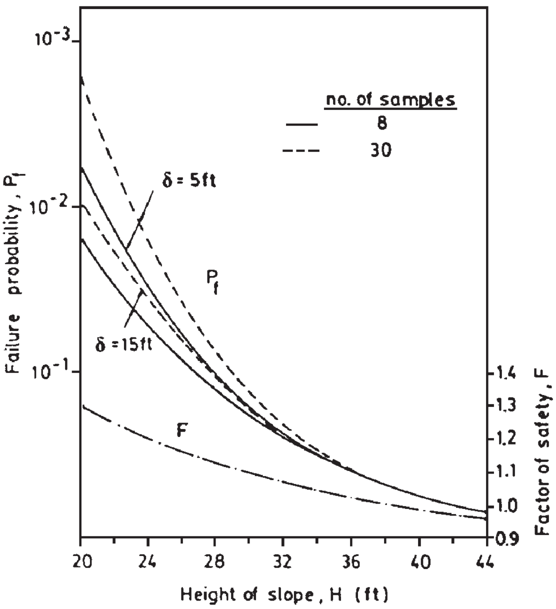

Figure 10. Variation of failure probability with height of slope (Selset Landslide) (1 ft = 0.305 m)

DISCUSSION

What has been presented here is a general probabilistic approach of slope design based on a two-dimensional limit equilibrium stability model. However, the discussion will not be complete without mentioning the limitations of the present approach.

A two-dimensional stability model is used herein. This is equivalent to saying that soil properties are perfectly correlated in the transverse direction. The consequence of such an assumption remains a question of further inquiry. But no doubt a three-dimensional analysis, especially for c-ϕ slopes, will be much more complicated than a two-dimensional analysis both in terms of the formulation of the performance function and the generation of the covariance matrix of the spatial average soil properties. No detailed probabilistic study on three-dimensional soil slopes has yet been published in the literature for c-ϕ slopes. However, attempts have been made to analyze a three-dimensional $\phi = 0$ slope using a probabilistic approach (Vanmarcke 1977*b*, 1980).

In a limit equilibrium analysis, soils are assumed to be perfectly plastic materials. On this basis, the spatially averaged soil properties will be the pertinent parameters to be used in the analysis. However, for strain-softening soils, the effect of “brittle” failure cannot be overlooked. In a conventional deterministic analysis in which the soil properties are assumed to be constant, the yield zone always initiates at the location with the highest stress level. However, the picture will be somewhat different when it is looked at from a probabilistic point of view. Since soil properties vary from point to point within a slope, there may be a chance that the soil strength is very low at a location where the stress level is not the highest. Failure can well initiate from this point instead of the most highly stressed region. On the other hand, if it so happens that the soil strength is the lowest at the most highly stressed region, the yield zone may propagate catastrophically to the adjoining area, leading to a sudden failure of slope.

The spatial variability of soil has therefore two opposing consequences. On the one hand, the spatial variability reduces the variance of the average soil properties and hence the failure probability of slopes. On the other hand, spatial variability of soil will increase the likelihood of progressive failure, as failure can initiate at any location along the slip surface. Which effect will dominate depends on the strain-softening behaviour of the soil. At present, study on this topic is limited. Further discussion is given in Tang *et al*. (1985).

So far, discussion has been focused on the failure probability for a particular slip surface. In fact, there are infinitely many admissible slip surfaces, although the failure probability for each of them may differ. A slope should be considered as a system in series. Each component represents a feasible slip surface. Failure of any slip surface (component) will imply the failure of the slope (the system). The system failure probability P_{fs} of a slope is bounded by (Cornell 1967)

$$(P_f)_{max} \le P_{fs} \le 1 \qquad (33)$$

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where $(P_f)_{max}$ is the failure probability for the most critical slip surface. If high correlation exists between different components, the system failure probability will be close to the lower probability bound. Studies by Morlá Catalán (1974) indicate that the system failure probability of a slope is substantially different from the failure probability for the most critical slip surface for the normal range of correlation existing between different slip surfaces. However, gross assumptions have been made by Morlá Catalán (1974) in the analysis, the consequence of which has yet to be observed. More research needs to be done before any conclusive remarks can be made.

Although there are still problems that remain to be solved regarding the probabilistic modelling of soil behaviour and the methodology for reliability calculations, it is pedantic to delay the use of the available probabilistic methods, especially the FOSM method, for want of a complete probabilistic analysis. The main advantage of using a probabilistic approach is to provide an operational procedure by which the uncertainties of the design can be considered in the analysis. It also helps the engineer to quantify his experience by way of building up his knowledge on the values of the statistical parameters such as the COV or scales of fluctuations of the local soils. These judgemental values can always be updated and uncertainties sharpened when more information becomes available. Moreover, experience is more easily transmitted to an inexperienced engineer by conveying to him the likely values of the statistical parameters of the soil properties with which he can perform his own uncertainty analysis than by just telling him what magic number should be used for the factor of safety.

CONCLUSION

This paper outlines a procedure of probabilistic slope design using the FOSM method and Morgenstern and Price's method as the stability model. The procedure can be applied to the analysis of the stability of a general slip surface. The reliability index β_{HL} is also used in lieu of the conventional reliability index β. The former has the advantage of being an invariant index of risk measure. Values of β_{HL} can be directly compared even though they may be derived from different formats for the performance function.

The reliability index β_{HL} can also be thought of as a standardized safety measure that suitably summarizes the uncertainties involved in the analysis. On this basis, alternative designs can be compared directly, whereas the comparison of FOS is debatable. The reliability index β_{HL} can also be related nominally to the failure probability by Eq. (24).

For a given interslice force function, investigations show that the reliability index β^m_{HL} based on moment equilibrium is much less sensitive to λ than the reliability index β^f_{HL} based on force equilibrium. Results also indicate that the rigorous reliability index β^r_{HL} is not sensitive to the interslice force function used in the analysis. Another encouraging observation is the insensitivity of β^r_{HL} to the ACF used. The correlation structure of a soil property is adequately described by the scale of fluctuation of the property. However, the influence of this parameter on the reliability index is most significant.

The critical failure surfaces with the minimum FOS and reliability index are close. Consequently, the location of the critical slip surface with the minimum FOS can be used as a good initial trial location for the search for the critical slip surface with the minimum reliability index.

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APPENDIX. PARTIAL DERIVATIVES OF THE PERFORMANCE FUNCTIONS

The following abbreviations are used in subsequent expressions:

$$\zeta_i = \frac{1}{\dfrac{1}{\lambda f_i} - t_i m_i + \tan\alpha_i} \tag{Al}$$

$$\left.\begin{aligned} d_i &= 0, && i = 1 \\ d_i &= \frac{1}{\lambda}\left\{\frac{1}{f(x_{i-1})} - \frac{1}{f(x_i)}\right\}, && i > 1 \end{aligned}\right\} \tag{A2}$$

(a) Cohesion

$$\left.\begin{aligned} \frac{\partial G_m}{\partial \tilde{c}_i'} &= \left(\Delta x_i + \frac{\partial \Delta T_i}{\partial \tilde{c}_i'} t_i\right) m_i y_{m_i} \\ &\quad - \frac{\partial \Delta T_i}{\partial \tilde{c}_i'}\left(y_{m_i}\tan\alpha_i - x_{m_i}\right) \\ &\quad + \sum_{j=i+1}^{n} \frac{\partial \Delta T_j}{\partial \tilde{c}_i'}\left[\left(t_j m_j - \tan\alpha_j\right) y_{m_j} + x_{m_j}\right], \\ &\qquad\qquad i \neq n \\ \frac{\partial G_m}{\partial \tilde{c}_i'} &= \Delta x_n m_n y_{m_n}, \qquad i = n \end{aligned}\right\} \tag{A3}$$

$$\left.\begin{aligned} \frac{\partial G_m}{\partial \tilde{c}_i'} &= \left(\Delta x_i + \frac{\partial \Delta T_i}{\partial \tilde{c}_i'} t_i\right) m_i - \frac{\partial \Delta T_i}{\partial \tilde{c}_i'}\tan\alpha_i \\ &\quad + \sum_{j=i+1}^{n} \frac{\partial \Delta T_j}{\partial \tilde{c}_i'}\left(t_j m_j - \tan\alpha_j\right), \quad i \neq n \\ \frac{\partial G_m}{\partial \tilde{c}_i'} &= \Delta x_n m_n, \qquad i = n \end{aligned}\right\} \tag{A4}$$

and

$$
\left.
\begin{aligned}
&\frac{\partial \Delta T_j}{\partial \tilde{c}_i{}'} = \Delta x_i m_i \zeta_i, && j = i \\
&\frac{\partial \Delta T_j}{\partial \tilde{c}_i{}'} = \frac{\partial T_{j-1}}{\partial \tilde{c}_i{}'} d_j \zeta_j, && i < j < n \\
&\frac{\partial \Delta T_j}{\partial \tilde{c}_i{}'} = -\frac{\partial T_{n-1}}{\partial \tilde{c}_i{}'}, && j = n
\end{aligned}
\right\} \quad \text{(A5)}
$$

$$
\left.
\begin{aligned}
&\frac{\partial \Delta T_j}{\partial \tilde{c}_i{}'} = \frac{\partial \Delta T_i}{\partial \tilde{c}_i{}'}, && j = i \\
&\frac{\partial \Delta T_j}{\partial \tilde{c}_i{}'} = \frac{\partial T_{j-1}}{\partial \tilde{c}_i{}'} + \frac{\partial \Delta T_j}{\partial \tilde{c}_i{}'}, && i < j < n
\end{aligned}
\right\} \quad \text{(A6)}
$$

(b) ΔW_{i}

$$
\left.
\begin{aligned}
\frac{\partial G_m}{\partial \Delta W_i} &= \left(1 + \frac{\partial \Delta T_i}{\partial \Delta W_i}\right) t_i m_i y_{m_i} \\
&\quad - \left[\left(1 + \frac{\partial \Delta T_i}{\partial \Delta W_i}\right) y_{m_i} \tan\alpha_i - \frac{\partial \Delta T_i}{\partial \Delta W_i} x_{m_i}\right] \\
&\quad + \sum_{j=i+1}^{n} \frac{\partial \Delta T_j}{\partial \Delta W_i}\left[\left(t_j m_j - \tan\alpha_j\right) y_{m_j} + x_{m_j}\right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad i \neq n \\
\frac{\partial G_m}{\partial \Delta W_i} &= \left(t_n m_n - \tan\alpha_n\right) y_{m_n}, \qquad\qquad i = n
\end{aligned}
\right\} \quad \text{(A7)}
$$

$$
\left.
\begin{aligned}
\frac{\partial G_f}{\partial \Delta W_i} &= \left(1 + \frac{\partial \Delta T_i}{\partial \Delta W_i}\right) t_i m_i - \left(1 + \frac{\partial \Delta T_i}{\partial \Delta W_i}\right) \tan\alpha_i \\
&\quad + \sum_{j=i+1}^{n} \frac{\partial \Delta T_j}{\partial \Delta W_i}\left(t_j m_j - \tan\alpha_j\right), \quad i \neq n \\
\frac{\partial G_f}{\partial \Delta W_i} &= t_n m_n - \tan\alpha_n, \qquad\qquad i = n
\end{aligned}
\right\} \quad \text{(A8)}
$$

and

$$
\left.
\begin{aligned}
&\frac{\partial \Delta T_j}{\partial \Delta W_i} = (t_i m_i - \tan\alpha_i)\zeta_i, && j = i \\
&\frac{\partial \Delta T_j}{\partial \Delta W_i} = \frac{\partial T_{j-1}}{\partial \Delta W_i} d_j \zeta_j, && i < j < n \\
&\frac{\partial \Delta T_j}{\partial \Delta W_i} = -\frac{\partial T_{n-1}}{\partial \Delta W_i}, && j = n
\end{aligned}
\right\} \quad \text{(A9)}
$$

$$
\left.
\begin{aligned}
&\frac{\partial T_j}{\partial \Delta W_i} = \frac{\partial \Delta T_i}{\partial \Delta W_i}, && j = i \\
&\frac{\partial T_j}{\partial \Delta W_i} = \frac{\partial T_{j-1}}{\partial \Delta W_i} + \frac{\partial \Delta T_j}{\partial \Delta W_i}, && i < j < n
\end{aligned}
\right\} \quad \text{(A10)}
$$

$$
\left.
\begin{aligned}
&\frac{\partial G}{\partial \tilde{\gamma}_i} = A_i \frac{\partial G}{\partial \Delta W_i} \\
&\frac{\partial G}{\partial p_i} = \Delta x_i \frac{\partial G}{\partial \Delta W_i} \\
&\frac{\partial G}{\partial \Delta P_i} = \frac{\partial G}{\partial \Delta W_i}
\end{aligned}
\right. \quad \text{(A11)}
$$

(c) Pore-water pressure ratio

The pore-water pressure $\tilde{u}_i$ is expressed in terms of the pore-water pressure ratio r_i, $\tilde{u}_i \Delta x_i = r_i A_i \tilde{\gamma}_i$ where $\tilde{\gamma}_i$ is the mean soil density.

$$
\left.
\begin{aligned}
\frac{\partial G_m}{\partial r_i} &= \left(\frac{\partial \Delta T_i}{\partial r_i} - A_i \bar{\gamma}_i\right) t_i m_i y_{m_i} \\
&\quad - \frac{\partial \Delta T_i}{\partial r_i}\left(y_{m_i} \tan\alpha_i - x_{m_i}\right) \\
&\quad + \sum_{j=i+1}^{n} \frac{\partial \Delta T_j}{\partial r_i}\left[\left(t_j m_j - \tan\alpha_j\right) y_{m_j} + x_{m_j}\right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad i \neq n \\
\frac{\partial G_m}{\partial r_i} &= -A_n \bar{\gamma}_n t_n m_n y_{m_n}, \qquad\qquad i = n
\end{aligned}
\right\} \quad \text{(A12)}
$$

$$
\left.
\begin{aligned}
\frac{\partial G_f}{\partial r_i} &= \left(\frac{\partial \Delta T_i}{\partial r_i} - A_i \bar{\gamma}_i\right) t_i m_i - \frac{\partial \Delta T_i}{\partial r_i} \tan\alpha_i \\
&\quad + \sum_{j=i+1}^{n} \frac{\partial \Delta T_j}{\partial r_i}\left(t_j m_j - \tan\alpha_j\right), \quad i \neq n \\
\frac{\partial G_f}{\partial r_i} &= -A_n \bar{\gamma}_n t_n m_n, \qquad\qquad i = n
\end{aligned}
\right\} \quad \text{(A13)}
$$

and

$$
\left.
\begin{aligned}
&\frac{\partial \Delta T_j}{\partial r_i} = -A_i \bar{\gamma}_i t_i m_i \zeta_i, && j = i \\
&\frac{\partial \Delta T_j}{\partial r_i} = \frac{\partial T_{j-1}}{\partial r_i} d_j \zeta_j, && i < j < n
\end{aligned}
\right\} \quad \text{(A14)}
$$

$$
\left.
\begin{aligned}
&\frac{\partial T_j}{\partial r_i} = \frac{\partial \Delta T_i}{\partial r_i}, && j = i \\
&\frac{\partial T_j}{\partial r_i} = \frac{\partial T_{j-1}}{\partial r_i} + \frac{\partial \Delta T_j}{\partial r_i}, && i < j < n
\end{aligned}
\right\} \quad \text{(A15)}
$$

In the illustrative examples given in the text, r_{i} is assumed to be perfectly correlated, i.e., the pore-water pressure ratio for the whole soil mass is represented by a single variable, r_{u}. The derivative of the performance function with respect to r_{u} is simply given by $\partial G / \partial r_u = \sum_{i=1}^{n} \partial G / \partial r_i$ using the formulae given above.

(d) Coefficient of internal resistance

$$\left.\begin{aligned}\frac{\partial G_m}{\partial t_i} &= \left[(1+t_i\tan\alpha_i)t_i\frac{\partial \Delta T_i}{\partial t_i} + \left(\Delta T_i + \Delta W_i - r_i\left(A_i\bar{\gamma}_i\right)\right) - \tilde{c}_i'\Delta x_i\tan\alpha_i\right] \\ &\quad \times m_i^2 y_{m_i}\cos^2\alpha_i - \frac{\partial \Delta T_i}{\partial t_i}\left(y_{m_i}\tan\alpha_i - x_{m_i}\right) \\ &\quad + \sum_{j=i+1}^{n}\frac{\partial \Delta T_j}{\partial t_i}\left[\left(t_j m_j - \tan\alpha_j\right)y_{m_j} + x_{m_j}\right], \qquad i \neq n \\ \frac{\partial G_m}{\partial t_i} &= \left(\Delta W_n + T_b - T_{n-1} - r_n\left(A_n\bar{\gamma}_n\right) - \tilde{c}_n\Delta x_n\tan\alpha_n\right)m_n^2 y_{m_n}\cos^2\alpha_n, \qquad i = n\end{aligned}\right\} \tag{A16}$$

$$\left.\begin{aligned}\frac{\partial G_f}{\partial t_i} &= \left[(1+t_i\tan\alpha_i)t_i\frac{\partial \Delta T_i}{\partial t_i} + \left(\Delta T_i + \Delta W_i - r_i\left(A_i\bar{\gamma}_i\right)\right) - \tilde{c}_i'\Delta x_i\tan\alpha_i\right] \\ &\quad \times m_i^2\cos^2\alpha_i - \frac{\partial \Delta T_i}{\partial t_i}\tan\alpha_i \\ &\quad + \sum_{j=i+1}^{n}\frac{\partial \Delta T_j}{\partial t_i}\left(t_j m_j - \tan\alpha_j\right), \qquad i \neq n \\ \frac{\partial G_f}{\partial t_i} &= \left(\Delta W_n + T_b - T_{n-1} - r_n\left(A_n\bar{\gamma}_n\right) - \tilde{c}_n\Delta x_n\tan\alpha_n\right)m_n^2\cos^2\alpha_n, \qquad i = n\end{aligned}\right\} \tag{A17}$$

and

$$\left.\begin{aligned}\frac{\partial \Delta T_j}{\partial t_i} &= \left(\Delta W_i - r_i A_i\bar{\gamma}_i - \tilde{c}_i'\Delta x_i\tan\alpha_i\right)m_i^2\zeta_i\cos^2\alpha_i \\ &\quad + \left\{E_{i-1} - \frac{T_{i-1}}{\lambda f_i} + \left[\tilde{c}_i'\Delta x_i + \left(\Delta W_i - r_i A_i\bar{\gamma}_i\right)t_i\right]m_i - \left(\Delta Q_i + \Delta W_i\tan\alpha_i\right)\right\} \\ &\quad \times\zeta_i^2 m_i^2\cos^2\alpha_i, \qquad j = i \\ \frac{\partial \Delta T_j}{\partial t_i} &= \frac{\partial T_{j-1}}{\partial t_i}d_j\zeta_j, \qquad i < j < n\end{aligned}\right\} \tag{A18}$$

$$\frac{\partial T_j}{\partial t_i} = \frac{\partial \Delta T_i}{\partial t_i}, \quad j = i; \qquad \frac{\partial T_j}{\partial t_i} = \frac{\partial T_{j-1}}{\partial t_i} + \frac{\partial \Delta T_j}{\partial t_i}, \quad \mathrm{i < j < n} \tag{A19}$$

(e) ΔQ_i

$$\left.\begin{aligned}\frac{\partial G_m}{\partial \Delta Q_i} &= \frac{\partial \Delta T_i}{\partial \Delta Q_i}t_i m_i y_{m_i} - \left[y_{Q_i} + \frac{\partial \Delta T_i}{\partial \Delta Q_i}\left(y_{m_i}\tan\alpha_i - x_{m_i}\right)\right] + \sum_{j=i+1}^{n}\frac{\partial \Delta T_j}{\partial \Delta Q_i}\left[\left(t_j m_j - \tan\alpha_j\right)y_{m_j} + x_{m_j}\right], \\ &\qquad i \neq n \\ \frac{\partial G_m}{\partial \Delta Q_i} &= -y_{Q_n}, \qquad i = n\end{aligned}\right\} \tag{A20}$$

$$\left.\begin{aligned}\frac{\partial G_f}{\partial \Delta Q_i} &= \frac{\partial \Delta T_i}{\partial \Delta Q_i}\left(t_i m_i - \tan\alpha_i\right) - 1 \\ &\quad + \sum_{j=i+1}^{n} \frac{\partial \Delta T_j}{\partial \Delta Q_i}\left(t_j m_j - \tan\alpha_j\right), \\ &\qquad\qquad i \neq n \\ \frac{\partial G_f}{\partial \Delta Q_i} &= -1, \qquad\qquad i = n\end{aligned}\right\} \quad \text{(A21)}$$

and

$$\left.\begin{aligned}\frac{\partial \Delta T_j}{\partial \Delta Q_i} &= -\zeta_i, & j &= i \\ \frac{\partial \Delta T_j}{\partial \Delta Q_i} &= \frac{\partial T_{j-1}}{\partial \Delta Q_i} d_j \zeta_j, & i &< j < n\end{aligned}\right\} \quad \text{(A22)}$$

$$\left.\begin{aligned}\frac{\partial T_j}{\partial \Delta Q_i} &= \frac{\partial \Delta T_j}{\partial \Delta Q_i}, & j &= i \\ \frac{\partial T_j}{\partial \Delta Q_i} &= \frac{\partial T_{j-1}}{\partial \Delta Q_i} + \frac{\partial \Delta T_j}{\partial \Delta Q_i}, & i &< j < n\end{aligned}\right\} \quad \text{(A23)}$$

(f) End forces

$$\frac{\partial G_m}{\partial E_a} = y_a + \sum_{i=1}^{n} \frac{\partial \Delta T_i}{\partial E_a}\left[\left(t_i m_i - \tan\alpha_i\right) y_{m_i} + x_{m_i}\right] \quad \text{(A24)}$$

$$\frac{\partial G_f}{\partial E_a} = 1 + \sum_{i=1}^{n} \frac{\partial \Delta T_i}{\partial E_a}\left(t_i m_i - \tan\alpha_i\right) \quad \text{(A25)}$$

$$\left.\begin{aligned}\frac{\partial \Delta T_j}{\partial E_a} &= \zeta_1, & j &= 1 \\ \frac{\partial \Delta T_j}{\partial E_a} &= \frac{\partial T_{j-1}}{\partial E_a} d_j \zeta_j, & 1 &< j < n \\ \frac{\partial \Delta T_j}{\partial E_a} &= -\frac{\partial T_{n-1}}{\partial E_a}, & j &= n\end{aligned}\right\} \quad \text{(A26)}$$

$$\left.\begin{aligned}\frac{\partial T_j}{\partial E_a} &= \frac{\partial \Delta T_j}{\partial E_a}, & j &= 1 \\ \frac{\partial T_j}{\partial E_a} &= \frac{\partial T_{j-1}}{\partial E_a} = \frac{\partial \Delta T_j}{\partial E_a}, & 1 &< j < n\end{aligned}\right\} \quad \text{(A27)}$$

$$\frac{\partial G_m}{\partial T_a} = -x_a + \sum_{i=1}^{n} \frac{\partial \Delta T_i}{\partial T_a}\left[\left(t_i m_i - \tan\alpha_i\right) y_{m_i} + x_{m_i}\right] \quad \text{(A28)}$$

$$\frac{\partial G_f}{\partial T_a} = \sum_{i=1}^{n} \frac{\partial \Delta T_i}{\partial T_a}\left(t_i m_i - \tan\alpha_i\right) \quad \text{(A29)}$$

$$\left.\begin{aligned}\frac{\partial \Delta T_j}{\partial T_a} &= -\frac{1}{\lambda f_1}\zeta_1, & j &= 1 \\ \frac{\partial \Delta T_j}{\partial T_a} &= \frac{\partial T_{j-1}}{\partial T_a} d_j \zeta_j, & 1 &< j < n \\ \frac{\partial \Delta T_j}{\partial T_a} &= -\frac{\partial T_{n-1}}{\partial T_a}, & j &= n\end{aligned}\right\} \quad \text{(A30)}$$

$$\left.\begin{aligned}\frac{\partial T_j}{\partial T_a} &= 1 + \frac{\partial \Delta T_1}{\partial T_a}, & j &= 1 \\ \frac{\partial T_j}{\partial T_a} &= \frac{\partial T_{j-1}}{\partial T_a} + \frac{\partial \Delta T_j}{\partial T_a}, & 1 &< j < n\end{aligned}\right\} \quad \text{(A31)}$$

$$\frac{\partial G_m}{\partial E_b} = -y_b \quad \text{(A32)}$$

$$\frac{\partial G_f}{\partial E_b} = -1 \quad \text{(A33)}$$

$$\frac{\partial G_m}{\partial T_b} = \left(t_n m_n - \tan\alpha_n\right) y_{m_n} + x_{m_n} - x_b \quad \text{(A34)}$$

$$\frac{\partial G_f}{\partial T_b} = t_n m_n - \tan\alpha_n \quad \text{(A35)}$$

It is worth pointing out that once the values of the parameters are given, the derivatives can be calculated explicitly and successively using the equations given in this Appendix. The above formulae are applicable to the case where λ is not zero. If λ is zero, which corresponds to the case of simplified Bishop or Janbu analysis, ΔT_i is identically zero. Therefore, all the derivatives of ΔT_i and T_i are to be replaced by zero in the above expressions.